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Integrability in SFT and new representation of dispersionless KP tau-function

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Abstract: We are investigating the properties of vacuum and boundary states in the CFT of free bosons under the conformal transformation. We show that transformed vacuum (boundary state) is given in terms of tau-functions of dispersionless KP (Toda) hierarchies. Applications of this approach to string field theory is considered. We recognize in Neumann coefficients the matrix of second derivatives of tau-function of dispersionless KP and identify surface states with the conformally transformed vacuum of free field theory.

Keywords: String Field Theory, Integrable Hierarchies.

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Contents

1. Introduction

Integrability plays an important role in physics. It allows one to go beyond perturbation theory and to discover properties of one's systems which are usually inaccessible in any other way. That is why when one finds a hidden integrability in the problem it usually means a major breakthrough in it and promises many new and unexpected results.

Recently the integrability was discovered behind the dynamics of conformal maps [1]–[3]. Namely, it was shown that analytic curves on the plane can be parametrized by their so called *harmonic moments* (set of complex variables t_k, $k \geq 0$). These moments proved to be good coordinates in the space of such curves. Evolution of the curve with respect to varying one of those moments, while keeping the rest fixed, turned out to be described by dynamical flows of the dispersionless 2D Toda Lattice hierarchy. As it is well known all these flows commute with each other and therefore t_k are indeed well-defined coordinates in the space of analytic curves.

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In particular it was shown that one can associate the tau-function with the space of analytic curves. Conformal transformation from a curve to the unit circle is expressed in terms of second derivatives of this tau-function calculated at values of its arguments, which coincide with harmonic moments t_k of the curve at hand. This allowed the authors of [3] to introduce the concept of *tau-function of analytic curves.* In section 3.1 we provide an overview of some results from works [2, 3].

These ideas were applied to several problems in condensed matter: (e.g. laplacian growth [1], Quantum Hall effect [8]) and quantum field theory (solutions of WDVV equations [7]). In each of those cases integrability helped to obtain new results.

In this paper we are proposing a different realization of the idea of connection between integrability and conformal maps. We realize a tau-function as a state which is the conformal transform of the vacuum of conformal field theory (CFT) in two dimensions. This is done in the following way. If one realizes Fock space of CFT in terms of functions of infinitely many formal variables s_k and represents creation (annihilation) operators as multiplication on s_k, (differentiation with respect to s_k), then the vacuum is just a constant function. Conformal transformation of the plane induces the linear transformation of creation and annihilation operators of the theory, mixing them in general. Then the natural question arises: what is the new vacuum (defined with respect to the new annihilation operators) as a function of the same variables s_k and conformal transformation (which is parametrized by harmonic moments t_k of the curve mapped by this transformation to the unit circle)?

We show in section 3 that this function (called $B(\boldsymbol{s}|\boldsymbol{t})$ in the paper) is a generating function of the (holomorphic) second derivatives of the (logarithm of[1]) tau-function of analytic curve calculated at the point t_k. Also, considered as a function of s_k with t_k's fixed, this function $B(\boldsymbol{s}|\boldsymbol{t})$ itself is a (logarithm of) tau-function of dispersionless KP (dKP) hierarchy [15, 10, 16]. As a function of s_k it is quite simple, being quadratic in all variables. Nevertheless it is closely related to some other non-trivial tau-function in variables t_k, namely the tau-function of dKP F_{herm}, given by large N limit of hermitean one-matrix model. Using the specific homogeneity condition enjoyed by this function we can identify it with our $B(s_k = t_k|t)$.

This gives a new, much simpler and more intuitive free field realization of some tau-functions of dKP (compared to the fermion constructions of [4]) and has immediate applications.

The first application we consider is in the area of String Field Theory (SFT) [26]. The language adopted in the present paper is an adequate one in case of SFT and is indeed widely used there. One of the basic objects in the SFT construction are so-called "Neumann" coefficients. In their terms the interaction of the theory (the star-product) can be defined [21]–[25]. They are expressed in terms of conformal mapping of world-sheets of three interacting strings. Using Neumann coefficients one can also construct the *surface states* - states in (boundary) CFT associated with the given conformal transformation.

[1] In case of dispersionless hierarchies logarithm of tau-function is a more natural object than tau-function itself. Throughout this paper we will abuse the terminology by calling it just a tau-function.

The description of such states is an important problem of SFT as some of them correspond to D-branes in SFT [19].

We show in section 4 that the "Neumann coefficients" associated with the class of conformal transformations considered in this paper are nothing else but second derivatives of tau-function of dKP hierarchy. This implies a number of properties, in particular the algebraic relation between the elements of the Neumann matrix. Also we show that our constructed tau-function $B(\boldsymbol{s}|\boldsymbol{t})$ *is* a representation of the surface state and thus provides new geometrical interpretation for this object.

Finally, we show in section 5 that the proper generalization of our construction to the case of CFT of free scalar field with the Dirichlet boundary conditions allows to find a similar representation (in terms of conformally transformed boundary state instead of conformally transformed vacuum) for the tau-function $\mathcal{B}(\boldsymbol{s},\bar{\boldsymbol{s}})$ of dispersionless 2D Toda Lattice hierarchy (dToda) [16]. This tau-function is related to the large N limit of the so-called normal matrix model [17, 18].

This generalization, being quite straightforward at the first glance, is non-trivial. Homogeneity conditions for the normal matrix model differ from those for hermitean matrix model. Identification of $\mathcal{B}(t,\bar{t})$ with the "tau-function of analytic curves" is thus more complicated compared to the previous case of hermitean one-matrix model. We will address it elsewhere [9]. Right now we only mention that this construction is needed for applications to the CFT description of the excitation of Quantum Hall Droplet and to 2D string theory. Both these problems are known to be related to normal matrix model (see e.g. [8] and [12, 13]).

2. General setup

Consider free chiral scalar field $\phi(w)$ in two dimensions

$$\phi(w) = \phi_0 + b_0 \log w - \sum_{k=-\infty}^{\infty}{}' \left(\frac{b_k}{k\, w^k}\right). \tag{2.1}$$

This theory is a free CFT. Consider its current $J(w)$

$$J(w) = \partial\phi(w) = \sum_{k=-\infty}^{\infty} \frac{b_k}{w^{k+1}} \tag{2.2}$$

b_k obey the usual commutation relations:

$$[b_k,\, b_n] = k\,\delta_{k+n,0} \tag{2.3}$$

$J(w)$ is a primary operator of this CFT with the conformal dimension $\Delta = (1,0)$, therefore we know that for any conformal transformation $w(z)$:

$$\tilde{J}(z)dz = J(w)dw|_{w=w(z)}\,. \tag{2.4}$$

We represent $\tilde{J}(z)$ in the form analogous to (2.2):

$$\tilde{J}(z) = \sum_{n=-\infty}^{\infty} \frac{a_n}{z^{n+1}}\,. \tag{2.5}$$

Commutation relations for the a_k's are the same as for b_k's. We can extract modes of the operator $J(w)$ in the following way:[2]

$$b_n = \oint_\infty \frac{dw}{2\pi i} w^n J(w) \,. \tag{2.6}$$

By virtue of (2.4) this can be written as:

$$b_n = \oint_\infty \frac{dz}{2\pi i} [w(z)]^n \tilde{J}(z) \,. \tag{2.7}$$

Analogously to the equation (2.6) we know that

$$a_k = \oint_\infty \frac{dz}{2\pi i} z^k \tilde{J}(z) \,. \tag{2.8}$$

Expanding $w(z)$ as a power series in z in eq. (2.7) we get the linear transformation from b_k to a_n.

General form of $w(z)$ which we will consider in this paper is

$$w(z) = \frac{z}{r} + \sum_{k\geq 0}^{\infty} \frac{p_k}{z^k} \,. \tag{2.9}$$

It is *univalent* at infinity i.e. maps region around $z = \infty$ into the region around $w = \infty$ in the one-to-one manner. General transformation will have the following form:

$$b_n = C_{n,n} a_n + C_{n,n-1} a_{n-1} + \cdots + C_{n,0} a_0 + \sum_{k>0} C_{n,-k} a_{-k} \,, \qquad n > 0 \tag{2.10}$$

$$b_{-n} = \sum_{k=n}^{\infty} C_{-n,-k} a_{-k} \,, \quad n > 0 \,, \tag{2.11}$$

where, $C_{k,n}$ are function of coefficients of conformal transformation (2.9). For example, $C_{n,n} = r^{-n}$, $C_{n,n-1} = n\, p_0\, r^{-(n-1)}$, etc. General form of $C_{n,k}$ is given by

$$C_{n,k} = \oint_\infty \frac{dz}{2\pi i} z^{-k-1} \Big(w(z) \Big)^n \quad \forall\, n, k \,. \tag{2.12}$$

In particular, $C_{0,k} = \delta_{0,k}$, which means that operator b_0 does not transform ($b_0 = a_0$).

Given set of operators b_k with the commutation relation (2.3) one can build a Fock space $\mathcal{F}_b$ with the vacuum, chosen by the condition (2.13):

$$b_n \left|0\right\rangle_b = 0, \; n \geq 0 \tag{2.13}$$

and b_{-n}'s acting on it as raising operators and thus building the $\mathcal{F}_b$. Correspondingly we may build a Fock space $\mathcal{F}_a$, starting from vacuum $\left|0\right\rangle_a$ and acting on it with operators a_{-k}'s.

Now, the question we want to ask is the following: what kind of transformation in the Fock space is induced by the transformation (2.10)–(2.11)? For example, what would correspond in the space $\mathcal{F}_a$ to the vacuum state (2.13)?

[2] Everywhere in this paper we choose orientation of the contour of integration such that $\oint_\infty dz\, z^{-1} = 2\pi i$

To answer this question we would like to pick explicit realization of Fock space $\mathcal{F}_a$. Let's realize the operators a_k in the following way:[3]

$$a_n = \frac{\partial}{\partial s_n}; \quad a_{-n} = n s_n; \quad n > 0\,. \tag{2.14}$$

Then Fock space $\mathcal{F}_a$ is the space of functions of infinitely many variables $s_1, s_2, \ldots$ Vacuum $|0\rangle_a$ is a constant.

Let's find the function, that corresponds in $\mathcal{F}_a$ to the vacuum (2.13) understanding operators b_k in terms of (2.10)–(2.11). This function (which we will look for in the form $\exp(B(s))$) obeys the system of equations:

$$\begin{aligned}
\frac{1}{r}\frac{\partial B(\boldsymbol{s})}{\partial s_1} &= -(p_1 s_1 + 2p_2 s_2 + 3p_3 s_3 + \cdots) \\
\frac{1}{r^2}\frac{\partial B(\boldsymbol{s})}{\partial s_2} &= -\left(\frac{2p_0}{r}\frac{\partial B(\boldsymbol{s})}{\partial s_1} + 2\left(p_0 p_1 + \frac{p_2}{r}\right) s_1 + 2\left(p_1^2 + 2p_0 p_2 + \frac{2p_3}{r}\right) s_2 + \cdots\right) \\
&\cdots\,.
\end{aligned} \tag{2.15}$$

One can easily integrate equations (2.15) to get (up to the constant of integration):

$$B(\boldsymbol{s}) = \frac{1}{2}\sum_{n,k=1}^{\infty} s_k s_n \frac{\partial^2 B(\boldsymbol{s})}{\partial s_k \partial s_n}\,, \tag{2.16}$$

where coefficients $\partial_{s_k}\partial_{s_n} B$ *do not depend* on s_k and are expressed in terms of r, p_k - coefficients of conformal map (2.9). For example one has:

$$\frac{1}{r}\frac{\partial^2 B(\boldsymbol{s})}{\partial s_1 \partial s_k} = -k p_k\,. \tag{2.17}$$

We put the constant of integration in (2.16) equal to zero.

3. Interpretations of function $B(\boldsymbol{s})$

In the sections that follow we are going to show that the function $B(s)$ can be identified with the logarithm of the tau-function of dispersionless KP hierarchy [10, 16, 15]. Note that the function $B(s)$ depends not only on the formal variables s_k (which correspond to creation operators a_{-k}), but also on the (coefficients of) conformal transformation $w(z)$. It is necessary for the future interpretation to describe this dependence explicitly. To do this we will need some basic information about the "good coordinates" in the space of conformal transformations (or, equivalently, in the space of analytic curves).

[3] We do not define the action of operator a_0 in this section, because we will be working in the subspace of $\mathcal{F}_a$, where $a_0\,|\psi\rangle = 0$

3.1 Digression about the tau-function of analytic curves

It was shown in [1]–[3] that one can describe the conformal mappings in the following convenient way. Consider mapping $w(z)$ from the exterior of the curve $\mathcal{C}$ in the z plane to the exterior of the unit circle in the w plane, univalent at infinity. General map of this type has the form (2.9). Obviously, map $w(z)$ fully describes the curve $\mathcal{C}$. Another way to parameterize the curve $\mathcal{C}$ is by the set of so-called *harmonic moments* t_k given by

$$t_k = \frac{1}{k}\oint_{\mathcal{C}} \frac{dz}{2\pi i}\bar{z}\, z^{-k}\,,\ k>0\,;\qquad t_0 = \oint_{\mathcal{C}} \frac{dz}{2\pi i}\bar{z}\,. \tag{3.1}$$

Set $t_0, t_1, \ldots$ (which we will collectively denote by $\boldsymbol{t}$) plays the role of coordinates in the space of the analytic curves. This means, that the coefficients of conformal map (2.9) are actually the functions of them: $r = r(\boldsymbol{t})$, $p_k = p_k(\boldsymbol{t})$. Explicit connection between those two sets of data for $\mathcal{C}$ is given by the function $F(\boldsymbol{t})$ — so called *tau-function of analytic curves* [3].

One of the definitions of the function $F(\boldsymbol{t})$ is

$$F(\boldsymbol{t}) = \frac{1}{\pi^2}\int\!\!\int_{\mathrm{int}\,\mathcal{C}} \log\left|\frac{1}{z}-\frac{1}{\zeta}\right| d^2z\, d^2\zeta\,. \tag{3.2}$$

This is a functional that maps space of analytic curves into the complex numbers. One can consider it as a function of t_k. This function turns out to be a logarithm of tau-function of dispersionless 2D Toda Lattice hierarchy (dToda). This is *the same* tau-function, that can be obtained in the large N limit of normal matrix model [17]. Note, that as a function of complex variables t_k $F(\boldsymbol{t})$ is usually *not* analytic, therefore we will think of it as an analytic function of two sets of variables (plus t_0): $F(\boldsymbol{t}) = F(t_0;\{t_k\};\{\bar{t}_k\})$. Nevertheless we will be using the notation $F(\boldsymbol{t})$.

Coefficients p_k's as functions of $\boldsymbol{t}$ can be read off the following useful relation:

$$\log\frac{w(z)}{z/r} = -\partial_{t_0} D(z) F(\boldsymbol{t})\,, \tag{3.3}$$

where

$$D(z) = \sum_{k=1}^{\infty}\frac{z^{-k}}{k}\frac{\partial}{\partial t_k} \tag{3.4}$$

and $\log r^2 = \partial_{t_0}^2 F(\boldsymbol{t})$.

As any tau-function, $F(\boldsymbol{t})$ obeys the set of *Hirota identities*. Hirota identities for dToda can be written in the form [3]:

$$D(z)D(\zeta)F(\boldsymbol{t}) - \frac{1}{2}\partial_{t_0^2}^2 F(\boldsymbol{t}) = \log\frac{w(z)-w(\zeta)}{z-\zeta} \tag{3.5}$$

which in view of (3.3) provides the relation between the second derivatives $\partial_{t_k}\partial_{t_n}F$ $(k,n>0)$ and $\partial_{t_0}\partial_{t_l}F$ and one can actually express *all* second derivatives $\partial_{t_k}\partial_{t_n}F$ in terms of the $\partial_{t_0}\partial_{t_l}F$ (see e.g. [15] and discussion in [7]). For the discussion below we would prefer to rewrite (3.3) and (3.5) in the form excluding any reference to the conformal transformation $w(z)$:

$$(z-\zeta)e^{D(z)D(\zeta)F} = ze^{-\partial_{t_0}D(z)F} - \zeta e^{-\partial_{t_0}D(\zeta)F}\,. \tag{3.6}$$

It should be noted that *any* tau-function of dToda hierarchy is also a tau-function of dKP hierarchy, considered as a function of $t_1, t_2, t_3, \ldots$ only with all other "times" (i.e. t_0, $\bar{t}_k$) fixed (c.f. [16]). The Hirota identity, which suits better for KP (and for the purpose of our paper) can be easily derived from (3.5). Taking $\zeta \to \infty$ in (3.5) one gets:

$$rw(z) = z + p_0 r - \partial_{t_1}D(z)F = z + p_0 r - \sum_{k=1}^{\infty} \frac{\partial^2 F(\boldsymbol{t})}{\partial t_1 \partial t_k}\frac{z^{-k}}{k} \tag{3.7}$$

and thus

$$\exp(D(z)D(\zeta)F) = 1 - \frac{D(z)\partial_{t_1}F - D(\zeta)\partial_{t_1}F}{z-\zeta}\,. \tag{3.8}$$

Eq. (3.8) is precisely the Hirota equation for dispersionless KP hierarchy [15, 16].

Note, that Hirota equations (3.8) (correspondingly (3.6)) are valid for *any* tau-function of the dKP (correspondingly dToda) hierarchy, not only for the "tau-function of analytic curves" described above. So, one can take point of view in a sense opposite to that of [2, 3]. One can *define* some univalent conformal map by the (3.7) for KP or (3.3) for Toda case, for *any* tau-function of the corresponding hierarchy.[4] In this case one would (in general) lose the interpretation of the times of dKP (dToda) tau-functions (let's call them $\tilde{t}_k$) as harmonic moments of the curve mapped to the unit circle.[5] by $w(z)$. However $\tilde{t}_k$ would still be coordinates in the space of conformal maps. The parameterization of the univalent conformal maps $w(z|\tilde{t}_k)$ is then given by the (3.7) or (3.3). If one uses only Hirota identities (as we will do in the following sections), then this parameterization of the conformal maps, given by different tau-function, can be used as well. However, anything which relies on the special properties of particular tau-function (which are often derived using geometrical interpretation, like (3.2)) would not be available any more. Example of such an interpretation for the dKP tau-function, related to the one-matrix model, is discussed in [6]. In their case the contour gets shrunk to the cut along the real axis and in this limit the tau-function of analytic curves goes into the partition sum of the hermitean one-matrix model.

3.2 Identification of second derivatives of function $B(s)$

We would like to re-express derivatives $\partial_{s_k}\partial_{s_n}B$ in terms of function $F(\boldsymbol{t})$, because this would clarify for us the meaning of function $B(s)$. The first hint that this expression can be very simple is given by comparison of eq. (3.7) with eq. (2.17). Indeed, comparing

[4]Of course, it is not obvious that for any tau-function $w(z)$, formally defined in this way, will have non-zero radius of convergence around infinity. But for the wide class of tau-functions it is so. Below when we say "arbitrary" tau-function we will mean "any tau-function, defining non-trivial conformal map via (3.7) or (3.3)".

[5]*Different* (slightly more involved) geometrical interpretation of the arguments of tau-function $\tilde{t}_k$ exists for *all* tau-functions [5].

coefficients in front of z^{-k} in the right and left hand sides of eq. (3.7) one can easily see that

$$\frac{\partial^2 F(\boldsymbol{t})}{\partial t_1 \partial t_k} = -krp_k, \quad \text{and thus} \quad \frac{\partial^2 F(\boldsymbol{t})}{\partial t_1 \partial t_k} = \frac{\partial^2 B(\boldsymbol{s})}{\partial s_1 \partial s_k}. \tag{3.9}$$

The natural guess would be that all second derivatives $\partial_{s_k}\partial_{s_n} B$ are equal to $\partial_{t_k}\partial_{t_n} F$. This is indeed the case, as we show in appendix A.

Thus function $B(\boldsymbol{s})$ becomes a *generating function* of the matrix of second derivatives: $\partial_{t_i}\partial_{t_j} F$:

$$B(\boldsymbol{s}|\boldsymbol{t}) = \frac{1}{2}\sum_{k,n=1}^{\infty} s_k s_n \frac{\partial^2 F(\boldsymbol{t})}{\partial t_k \partial t_n}. \tag{3.10}$$

To avoid confusion let's stress once again, that function $B(\boldsymbol{s}|\boldsymbol{t})$ is quadratic function in s_k and its dependence on parameters t_k is defined entirely by $\partial^2_{t_k t_n} F$ and this fact is express by notations (3.10).

Let us also mention here, that we did not use yet any properties of the particular tau-function $F(\boldsymbol{t})$ defined in [2, 3]. In appendix A we only used the fact that tau-function satisfies Hirota equations (3.5) for the $w(z)$ satisfying (3.3). The only place where we implicitly supposed that F is the "tau-function of analytic curves" is where we think about t_k as being harmonic moments of the curve, mapped by $w(z)$ to the unit circle, i.e. we use the particular parameterization of conformal maps $w(z|t_k)$. As it was stressed in the previous section (see also footnote 5, p. 5), in principle we could use equivalent description in terms of different tau-function $\tilde{F}(\tilde{t}_k)$ which would define for us different parameterization of the conformal maps $w(z|\tilde{t}_k)$ given again by the same formulae (3.3) (or (3.7)). As a result we would also get an equivalent to (3.10) formula for $B(\boldsymbol{s}|\boldsymbol{t})$ (which in this case we call $B(\boldsymbol{s}|\{w\})$ to stress once again its dependence on the conformal mapping $w(z)$ and not on particular parameterization thereof)

$$B(\boldsymbol{s}|\{w\}) = B(\boldsymbol{s}|\tilde{t}_k) = \frac{1}{2}\sum_{k,n=1}^{\infty} s_k s_n \frac{\partial^2 \tilde{F}(\tilde{t})}{\partial \tilde{t}_k \partial \tilde{t}_n}. \tag{3.11}$$

An example of such an equivalent description, which will be of interest for us here is the one for $\tilde{F} = F_{\text{herm}}$ and $\tilde{t}_k = T_k$. Here F_{herm} is the tau-function of the dKP hierarchy, equal to the large N limit of the partition sum of the hermitean one-matrix model and T_k are new moments (coupling constants of the matrix model), defined in [6]. Then we can rewrite $B(\boldsymbol{s}|\{w\})$ in the form

$$B(\boldsymbol{s}|\{w\}) = B(\boldsymbol{s}|T_k) = \frac{1}{2}\sum_{k,n=1}^{\infty} s_k s_n \frac{\partial^2 F_{\text{herm}}(T)}{\partial T_k \partial T_n}. \tag{3.12}$$

In fact, applying the limiting procedure of [6] one can obtain representation (3.12) for B directly from (3.10).

3.3 Function $B(\boldsymbol{s}|\boldsymbol{t})$

The representation (3.10) means that function $B(\boldsymbol{s}|\boldsymbol{t})$ (as a function of s_k with all t_k being held constant) satisfies Hirota equation (3.8) and thus by itself is the logarithm of tau-function of dKP hierarchy.

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Thus set of equations (2.15) may be considered as another form of Hirota identities, because it allows to express all second derivatives of logarithm of tau-function of dKP in terms of derivatives with respect to t_1, t_k. In the section 4 we will find yet another interpretation of this function.

One may ask the question then: "What is the particular condition, which selects this tau-function among all other tau-functions of dispersionless KP hierarchy?". Let us note, that everything which is said above in this section could be valid for any parameterization of the conformal maps given by some tau-function as it was discussed before. The function $B(\boldsymbol{s})$ as a function of s_k only will not change if we change this parameterization and the tau-functions $\tilde{F}(\tilde{t})$. Note also that equation (2.16) looks similar to the equation

$$\frac{1}{2}\sum_{k,n=1}^{\infty} t_k t_n \frac{\partial^2 F^{(0)}_{\mathrm{herm}}}{\partial t_k \partial t_n} = F^{(0)}_{\mathrm{herm}}\,, \tag{3.13}$$

where $F^{(0)}_{\mathrm{herm}}$ (called F_{herm} in the section 3.2) is the leading term of the partition sum of hermitean one-matrix model in the large N limit (c.f. appendix C). Eq. (3.13) is the consequence of the homogeneity condition which is obeyed by particular tau-function of KP, given by this matrix model (see App. C for details). Thus we are able to identify our tau-function $B(\boldsymbol{s}|T)$ with the one, given by hermitean one-matrix model in large N limit:

$$B(T|T) = \frac{1}{2}\sum_{k,n=1}^{\infty} T_k T_n \frac{\partial^2 F_{\mathrm{herm}}(T)}{\partial T_k \partial T_n} = F_{\mathrm{herm}}(T)\,. \tag{3.14}$$

4. String field theory

Let's turn to the application of these ideas now. As mentioned before, the language of this paper is useful in the completely different field of String Field Theory (SFT) (c.f. [21, 22]). The subject is huge and there are many reviews of it (see, e.g. [20] and references therein). Here we briefly remind necessary for us formulae. This is not intended as an introduction to the subject, but only serves to specify our notations.

Action of the *Cubic String Field Theory* has the following schematic form [26]:

$$S_{SFT} = \frac{1}{2}\int \Phi * Q_B \Phi + \frac{1}{3}\int \Phi * \Phi * \Phi\,. \tag{4.1}$$

We will not discuss the kinetic term here. Cubic vertex

$$V(A,B,C) \equiv \int \Phi_A * \Phi_B * \Phi_C \tag{4.2}$$

can be defined in several ways. The first (so called *operator*) approach is the following: given three string states $|A\rangle_1, |B\rangle_2, |C\rangle_3$ (corresponding to Φ_A, Φ_B, Φ_C), each belonging to its own Hilbert space $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$, their interaction vertex $V(A,B,C)$ is given by

$$V(A,B,C) = \langle V_3| \Big(\,|A\rangle \otimes |B\rangle \otimes |C\rangle \Big)\,, \tag{4.3}$$

where $\langle V_3| \in \mathcal{H}_1^* \otimes \mathcal{H}_2^* \otimes \mathcal{H}_3^*$ is defined[6] by the expression:

$$\langle V_3| = \langle 0|_1 \otimes \langle 0|_2 \otimes \langle 0|_3 \exp\left(-\frac{1}{2}\sum_{r,s=1}^{3}\sum_{n,m=1}^{\infty} \alpha_n^{(r)} N_{nm}^{rs} \alpha_m^{(s)} \right) . \tag{4.4}$$

Here $\alpha_m^{(s)}$ are modes of the scalar field, indices $r,s = 1,2,3$ number Hilbert spaces $\mathcal{H}_r$ of each of three strings and *Neumann coefficients* N_{nm}^{rs} are given by

$$N_{nm}^{rs} = \frac{1}{n\,m}\oint_0 \frac{dz}{2\pi i} z^{-n} \oint_0 \frac{d\zeta}{2\pi i}\zeta^{-m} \frac{f_r'(z) f_s'(\zeta)}{(f_r(z) - f_s(\zeta))^2} , \tag{4.5}$$

where $f_r(z)$ are the conformal transformations of the upper-half plane of each of the strings to the unit circle. In terms of $|V_3\rangle$ one can also define a star-multiplication of any two states:

$$|A * B\rangle = \Big(\langle A| \otimes \langle B| \Big) |V_3\rangle \ . \tag{4.6}$$

There exists another definition of $V(A,B,C)$ more useful in applications [21, 22]:

$$V(A,B,C) = \langle (f_1 \circ \Phi_A)(0)\, (f_2 \circ \Phi_B)(0)\, (f_3 \circ \Phi_C)(0) \rangle , \tag{4.7}$$

where correlator in the r.h.s. of (4.7) is computed in any CFT (e.g. CFT on the unit disk) and the maps f_i $(i = 1,2,3)$ are the same as in eq. (4.5). Expression in the r.h.s. of (4.7) has the following meaning. One takes (primary) operator Φ and acts on it with the conformal transformation $f(z)$ (we will denote the conformal image of Φ under the action of f as $(f \circ \Phi)$ or $f[\Phi]$):

$$(f \circ \Phi)(z) = f[\Phi(z)] = (f'(z))^d \Phi(f(z)) = U_f \Phi(z) U_f^{-1} , \tag{4.8}$$

where d is the conformal dimension of the operator. The operator U_f can be realized in the following way (for $f(z)$ regular at the origin):[7]

$$U_f = \exp\left(\sum_{n \geq 2} v_n L_n \right) , \tag{4.9}$$

where v_n are Laurent modes of the function $v(z) = \sum v_n z^{n+1}$, related to the $f(z)$ in the following way:

$$e^{v(z)\partial_z} z = f(z) . \tag{4.10}$$

One can associate the state $\langle f|$, corresponding to any conformal map $f(z)$, defined as

$$\langle f|\, \Phi\rangle = \langle 0|\, (f \circ \Phi)\, |0\rangle \quad \text{for all operators } \Phi . \tag{4.11}$$

In particular:

$$\langle f| \ldots \alpha_{k_1} \ldots \alpha_{k_2} \ldots |0\rangle = \langle 0| \ldots f[\alpha_{k_1}] \ldots f[\alpha_{k_2}] \ldots |0\rangle \ , \tag{4.12}$$

[6]We write eq. (4.4) schematically, for one scalar field, suppressing ghosts, integration over momenta, etc. (for details see e.g. [23, 24, 25] or any review in SFT)

[7]We have as usual $L_n\,|0\rangle = 0,\ n \geq -1$ and correspondingly $\langle 0|\, L_n = 0,\ n \leq 1$.

where α_k are modes of the expansion of the scalar field $X(z)$ and vacuum $|0\rangle$ is defined with respect to them. The state $\langle f|$ can be represented as

$$\langle f| = \langle 0|\, U_f\,. \tag{4.13}$$

Indeed, notice that

$$U_f^{-1}\,|0\rangle = U_f\,|0\rangle = |0\rangle \tag{4.14}$$

for U_f given by (4.9) and taking into account footnote 7, p. 10 we get the result (4.12). For the purpose of searching surface states one usually needs to find coefficients v_n, thus one needs to solve the equation (4.10), which is quite non-trivial generally.

State (4.13) has oscillator representation as well. Let's consider free scalar field $X(z)$. Then

$$\langle f| = \langle 0| \exp\left(-\frac{1}{2}\sum_{n,m=1}^{\infty} \alpha_n N^f_{nm}\alpha_m\right). \tag{4.15}$$

Coefficients N^f_{nm} are given by the analog of (4.5):

$$N^f_{nm} = \frac{1}{n\,m}\oint_0 \frac{dz}{2\pi i} z^{-n}\oint_0 \frac{d\zeta}{2\pi i}\zeta^{-m}\frac{f'(z)f'(\zeta)}{(f(z)-f(\zeta))^2}\,. \tag{4.16}$$

Representation (4.15) is easy to derive if one considers only two operators α_k, α_n in eq. (4.12) and notices that integrand in the r.h.s. of eq. (4.16) is just a correlator of $\langle f[\partial X(z)] f[\partial X(\zeta)]\rangle$.

4.1 Neumann coefficients as second derivatives of tau-function

Now we would like to repeat this construction for the case at hand. Namely, we will consider $w(z)$, given by (2.9) instead of $f(z)$ and construct the corresponding surface state $|w\rangle$.

First, let's introduce the field $\boldsymbol{v}(z)$:

$$e^{\boldsymbol{v}(z)\partial_z} z \equiv w(z)\,. \tag{4.17}$$

As $w(z)$ is regular at infinity, the expansion of this field $\boldsymbol{v}(z) = \sum \boldsymbol{v}_n z^{n+1}$ has only $n \le 1$ modes non-zero. As a result U_w, corresponding to eq. (4.9) is given by

$$U_w = \exp\left(\sum_{n\le -1} \boldsymbol{v}_n L_n\right) \tag{4.18}$$

and contrary to the property $U_f\,|0\rangle = |0\rangle$ here we have

$$\langle 0|\, U_w = \langle 0|\, U_w^{-1} = \langle 0|\ . \tag{4.19}$$

Let's apply the transformation U_w in the operators in the $\mathcal{F}_a$. We get

$$(w\circ \tilde{J})(z) = U_w \tilde{J}(z) U_w^{-1}\,. \tag{4.20}$$

So, for $U_w a_k U_w^{-1} \equiv w[a_k]$ as in previous section, we can define surface state $|w\rangle$

$$\langle 0| \ldots a_{k_1} \ldots a_{k_2} \ldots |w\rangle \; = \; \langle 0| \ldots w[a_{k_1}] \ldots w[a_{k_1}] \ldots |0\rangle \ , \tag{4.21}$$

where

$$|w\rangle \; = U_w^{-1} \, |0\rangle \; = \exp\left(\frac{1}{2} \sum_{n,m=1}^{\infty} a_{-n} N_{nm}^{w} a_{-m} \right) |0\rangle \ . \tag{4.22}$$

Using the definition (4.20) and operator product expansion for the current $\tilde{J}(z)$ one can show (see e.g. [22]) that Neumann coefficients N_{nm}^{w} here are given by:

$$N_{nm}^{w} = \frac{1}{n\,m} \oint_{\infty} \frac{dz}{2\pi i} z^n \oint_{\infty} \frac{d\zeta}{2\pi i} \zeta^m \frac{w'(z) w'(\zeta)}{(w(z) - w(\zeta))^2} \, . \tag{4.23}$$

One can rewrite this expression in the following form:

$$N_{nm}^{w} = \frac{1}{n\,m} \oint_{\infty} \frac{dz}{2\pi i} z^n \oint_{\infty} \frac{d\zeta}{2\pi i} \zeta^m \partial_z \partial_\zeta \log\left(\frac{w(z) - w(\zeta)}{z - \zeta} \right) . \tag{4.24}$$

Note, that

$$\oint_{\infty} \frac{dz}{2\pi i} z^{-n} \oint_{\infty} \frac{d\zeta}{2\pi i} \zeta^{-m} \frac{1}{(z-\zeta)^2} = 0 \qquad \forall \, n, m > 0 \, . \tag{4.25}$$

Comparing equation (4.25) with that of (3.5) we come to the conclusion that

$$N_{nm} = \frac{1}{n\,m} \frac{\partial^2 F(\boldsymbol{t})}{\partial t_n \partial t_m} \, . \tag{4.26}$$

This means that Neumann matrix is actually the matrix of second derivatives of one function, $F(\boldsymbol{t})$, which is associated with conformal map $w(z)$ in the way described in the section 3.1!

This fact in particular provides a number of relation between the matrix elements of N_{nm} (see e.g. [15, 7]) as a consequence of eq. (3.8). We show first several of them:

$$\begin{aligned}
N_{22} &= N_{13} - \frac{1}{2} N_{11}^2 \\
N_{23} &= N_{14} - N_{11} N_{12} \\
N_{33} &= \frac{1}{3} N_{11}^3 - N_{11} N_{13} - N_{12}^2 + N_{15} \\
&\ldots
\end{aligned}$$

Of the whole matrix N_{nm} only coefficients N_{1k} are independent!

We should also mention that this construction is trivially generalized for the case of conformal transformations, regular at the origin $z = 0$, considered in section 4 (see e.g. [6]). In this case Neumann coefficients (4.16) are expressed by the same equation (4.26) with *the same* tau-function F.

4.2 Surface state as conformally transformed vacuum $B(\boldsymbol{s})$

If the representation (2.14) for operators a_{-n} is used, surface state (4.22) looks identical to the exponential of (3.10). Indeed, one can see that "conformally transformed vacuum" $|0\rangle_b$ coincides with (4.22), i.e.

$$b_k(a)\,|w\rangle\, = 0\,. \tag{4.27}$$

To see this, note, that expressions $U_w a_k U_w^{-1} \equiv w[a_k]$ are *not* equal to b_k. Expressions for b_k are given by the inverse transformation

$$b_k = U_{w^{-1}} a_k U_{w^{-1}}^{-1} = U_w^{-1} a_k U_w\,. \tag{4.28}$$

Eq. (4.28) is identical to (2.7) or equivalently (2.10)–(2.11) i.e. b_k are just $w^{-1}[a_k]$. From this and the first formula of (3.10) one can easily see that $b_k\,|w\rangle\, = (U_w^{-1} a_k U_w)\; U_w^{-1}\,|0\rangle\, = U_w^{-1}\; a_k\,|0\rangle\, = 0$.

We see that the two constructions give equivalent definitions of $|w\rangle$ and "surface state" is nothing else but "conformally transformed vacuum" which was identified in the previous sections with the quadratic tau-function of integrable hierarchy $B(\boldsymbol{s}|\boldsymbol{t})$(or, equivalently, generating function of second derivatives of the tau-function corresponding to the matrix model).

5. CFT with the boundary and dToda tau-function

If function $B(\boldsymbol{s}|\boldsymbol{t})$ is actually a logarithm of tau-function of dKP hierarchy, the question arises — can tau-function for some other hierarchy be obtained in a similar way. We will demonstrate in this section that one can obtain the tau-function of dispersionless 2D Toda Lattice hierarchy in the way similar to that, taken in sections 2–3.

Consider the scalar field in the exterior of the unit circle in the plane w, with the Dirichlet boundary conditions on the circle:

$$\phi(w,\bar w)\; = b_0 \log |w|^2 - \sum_{k=-\infty}^{\infty}{}' \left(\frac{b_k}{k\,w^k} + \frac{\bar b_k}{k\,\bar w^k}\right) \tag{5.1}$$

$$\phi(w,\bar w)\Big|_{|w|=1} = 0\,. \tag{5.2}$$

Presence of the boundary makes holomorphic and anti-holomorphic modes dependent, which can be expressed via *boundary state* $|\beta\rangle$:

$$(b_k - \bar b_{-k})\,|\beta\rangle\, = 0\,, \qquad \forall\, k > 0\,. \tag{5.3}$$

Bringing together expressions (2.10) and (2.11) (and their analogs for the $\bar b_k$) we get

$$\sum_{n=1}^{k} C_{k,n} a_n + C_{k,0} a_0 + \sum_{n=k}^{\infty} C_{-k,n} a_{-n} - \sum_{n=k}^{\infty} \bar C_{-k,n} \bar a_{-n} = 0\,. \tag{5.4}$$

where $C_{k,n}$ is given by (2.12). Similarly

$$\bar C_{-k,n} = \oint_\infty \frac{dz}{2\pi i} z^{-n-1} \Big(\bar w(z)\Big)^{-k}\,, \qquad -\infty < n \le 0\,,\; k > 0\,. \tag{5.5}$$

Again, we realized a_k as in (2.14) and introduce new variables $\bar{s}_k$, in terms of which we would realize $\bar{a}_k$ similarly to (2.14), and s_0 in terms of which a_0 is realized as multiplication operator (note, that here, contrary to the case of section 2 boundary state (5.3) imposes no restriction on b_0). Then the conformally transformed state in the form $\exp\Big(\mathcal{B}(\boldsymbol{s},\bar{\boldsymbol{s}})\Big)$ obeys the equations

$$\sum_{n=1}^{k} C_{k,n}\frac{\partial \mathcal{B}}{\partial s_n} + C_{k,0}s_0 + \sum_{n=k}^{\infty} C_{-k,n} n s_n - \sum_{n=k}^{\infty} \bar{C}_{-k,n} n \bar{s}_n = 0\,, \qquad k>0 \tag{5.6}$$

and analogs of (5.6) where coefficients are conjugated and s_k interchanged with $\bar{s}_k$.

Again, one can show (see appendix B) that not only $\partial_{s_k}\partial_{s_n}\mathcal{B} = \partial_{s_k}\partial_{s_n}B = \partial_{t_k}\partial_{t_n}F$ but also mixed second derivatives $\partial^2_{s_k\bar{s}_n}\mathcal{B}$ actually coincide with $\partial_{t_k}\partial_{\bar{t}_n}F$. As a result, we get the generating function of the matrix of second derivatives (5.7) $\mathcal{B}(s_0,\boldsymbol{s},\bar{\boldsymbol{s}})$:

$$\begin{aligned}\mathcal{B}(s_0,\boldsymbol{s},\bar{\boldsymbol{s}}) = {}& \frac{s_0^2}{2}\frac{\partial^2 F}{\partial t_0^2} + \frac{1}{2}\sum_{k,n=1}^{\infty}\left(s_k s_n \frac{\partial^2 F}{\partial t_k \partial t_n} + \bar{s}_k\bar{s}_n \frac{\partial^2 F}{\partial \bar{t}_k \partial \bar{t}_n}\right) + \\ & + s_0\sum_{k=1}^{\infty}\left(s_k \frac{\partial^2 F}{\partial t_0 \partial t_k} + \bar{s}_k \frac{\partial^2 F}{\partial t_0 \partial \bar{t}_k}\right) + \sum_{k,n=1}^{\infty} s_k\bar{s}_n \frac{\partial^2 F}{\partial t_k \partial \bar{t}_n}\end{aligned} \tag{5.7}$$

here we've chosen "constant of integration" to be $\frac{s_0^2}{2}\frac{\partial^2 F}{\partial t_0^2}$, so that function $\mathcal{B}$ would be the generating function for *all* second derivatives of dispersionless 2D Toda tau-function.

5.1 Homogeneity condition for 2D Toda tau-function

Recall that in section 3.3 we noticed that function $B(\boldsymbol{s}|\boldsymbol{t})$ has the same scaling as partition sum of hermitean matrix model. This allowed us to identify $B(T|T) = F_{herm}(T)$. Let us see if something similar is possible in the present case. In the large N limit partition sum of normal matrix model obeys (see appendix D):

$$\sum_{k=1}^{\infty}\left(1-\frac{k}{2}\right)t_k\frac{\partial F_0}{\partial t_k} + \left(1-\frac{k}{2}\right)\bar{t}_k\frac{\partial F_0}{\partial \bar{t}_k} + t_0\frac{\partial F_0}{\partial t_0} = 2F_0\,. \tag{5.8}$$

We see that this homogeneity condition has very different than (5.7). Thus we can not repeat naively the trick like with the equation (3.14). This fact is not just a technical detail. It has important mathematical reason and consequences for physical interpretation. Detailed discussion of this issue is beyond the scope of the present paper. We would like only to note here that it is important for the application of the method developed here to the CFT description of the edge excitation of Quantum Hall Effect. We are going to return to it elsewhere [9].

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A. Computation for holomorphic derivatives

We are going to show that all second derivatives of the function $B(\boldsymbol{s})$ are expressed through the second derivatives of the function $F(\boldsymbol{t})$.

Taking n^{th} equation in (2.15) and differentiating it with respect to s_k one gets:

$$\sum_{m=1}^{n} C_{n,m} \frac{\partial^2 B}{\partial s_k \partial s_m} + k\, C_{n,-k} = 0\,. \tag{A.1}$$

We can rewrite it using definition of (2.12) as

$$\sum_{m=1}^{n} \oint_\infty \frac{dz}{2\pi i} \frac{w^n(z)}{z^{m+1}} \frac{\partial^2 B(\boldsymbol{s})}{\partial s_m \partial s_k} = -k \oint_\infty \frac{dz}{2\pi i} w^n(z) z^{k-1}\,, \qquad k,n>0\,. \tag{A.2}$$

Obviously, eq. (2.17) was just a particular case of (A.2) for $n=1$. We are going to substitute $\partial_{t_n}\partial_{t_m}F$ into equations (A.2) and show that they hold as a consequence of Hirota identity (3.5).

First, consider the l.h.s. of (A.2). Note that as a consequence of (2.9) we can substitute ∞ instead of n as an upper summation index in (A.2)

$$\sum_{m=1}^{n} \oint_\infty \frac{dz}{2\pi i} \frac{w^n(z)}{z^{m}} \frac{\partial^2 F(\boldsymbol{t})}{\partial t_m \partial t_k} = -\oint_\infty \frac{dz}{2\pi i} w^n(z) \partial_{t_k} D'(z) F(\boldsymbol{t})\,, \tag{A.3}$$

where we denoted by $D'(z) = \partial_z D(z)$, with operator $D(z)$ defined in (3.4). Then, taking derivative with respect to z of Hirota eq. (3.5) we get:

$$\partial_{t_k} D'(z) F = \oint_\infty \frac{d\zeta}{2\pi i} k\zeta^{k-1} \left(\frac{w'(z)}{w(z)-w(\zeta)} - \frac{1}{z-\zeta} \right) \tag{A.4}$$

(recall note 2, p. 4). We assume first that contour of integration in (A.4) is chosen so that $|z|>|\zeta|$. Then we can perform the integration over ζ in the last term of (A.4), which gives zero. Substituting expression (A.4) back into the r.h.s. of (A.3) we get

$$-\oint_\infty \frac{dz}{2\pi i} w^n(z) \partial_{t_k} D'(z) F(\boldsymbol{t}) = -\oint_\infty \frac{d\zeta}{2\pi i} k\zeta^{k-1} \left(\oint_\infty \frac{dw}{2\pi i} \frac{w^n}{w-w(\zeta)} \right). \tag{A.5}$$

We have chosen $|z|>|\zeta|$ which means that $|w(z)|>|w(\zeta)|$ and hence the contour of integration over w in (A.5) goes between poles at $w=\infty$ and $w=w(\zeta)$. The result of integration gives $w^n(\zeta)$ which together with (A.3) competes the proof.[8]

B. Computations for mixed derivatives

In case of dispersionless 2D Toda there is another type of Hirota identity, along with (3.5) — the mixed one:

$$1 - \exp(-D(z)\bar{D}(\bar{\zeta})F) = \frac{1}{z\bar{\zeta}} \exp(\partial_{t_0}(\partial_{t_0} + D(z) + \bar{D}(\bar{\zeta}))F)\,. \tag{B.1}$$

[8]If we have chosen $|z|<|\zeta|$, then the integral in (A.5) would give zero (both poles inside the contour), but the last integral in (A.4) would be equal to $k\zeta^{k-1}$ instead. So, the final result would be of course the same.

The first equation of Hirota (B.1) is

$$\frac{\partial^2 F}{\partial t_1 \partial \bar{t}_1} = \exp(\partial^2_{t_0} F)\,. \tag{B.2}$$

By differentiating (B.2) with respect to t_0 we get the first equation of dispersionless Toda hierarchy for the function u, such that $\partial_{t_0} u = \log r^2 = \partial^2_{t_0} F$:

$$\frac{\partial^2 u}{\partial t_1 \partial \bar{t}_1} = \frac{\partial}{\partial t_0} \exp\left(\frac{\partial u}{\partial t_0}\right)\,. \tag{B.3}$$

Eq. (B.1) expresses mixed derivatives in terms of derivatives with respect to t_0 and t_k or t_0 and $\bar{t}_k$. Now, consider set of equations (5.6). First of all, it is obvious that second derivatives with respect to s_k, s_n obey the same system of equations (A.1) and thus are derivatives of the function F with respect to the appropriate harmonic moments t_k, t_n. By differentiating eq. (5.6) with respect to s_0 one can re-write it in the following form:

$$\forall\, n > 0: \quad \sum_{k=1}^{n} \frac{\partial^2 \mathcal{B}(\boldsymbol{s}, \bar{\boldsymbol{s}})}{\partial s_0 \partial s_k} \oint_\infty \frac{dz}{2\pi i} w(z)^n z^{-k-1} + \oint_\infty \frac{dz}{2\pi i} w(z)^n z^{-1} = 0\,. \tag{B.4}$$

As in appendix A we would like to substitute $\partial_{s_0}\partial_{s_k}\mathcal{B}$ with $\partial_{t_0}\partial_{t_k} F$ in eq. (B.4) show that $\partial_{t_0}\partial_{t_k} F$ obey precisely the same set of equations, which can be written as (c.f. comment before eq. (A.3)):

$$\oint_\infty \frac{dz}{2\pi i} w(z)^n \partial_{t_0} D'(z) F = \oint_\infty \frac{dz}{2\pi i} w(z)^n z^{-1}\,, \tag{B.5}$$

where $D'(z)$ was define in appendix A. As a consequence of (3.3) we can write

$$-\,D'(z)\partial_{t_0} F = \frac{w'(z)}{w(z)} - \frac{1}{z}\,. \tag{B.6}$$

Multiplying l.h.s. of (B.6) by $w(z)^n$ and integrating around $z = \infty$ we get:

$$\oint_\infty \frac{dz}{2\pi i} w(z)^n \partial_{t_0} D'(z) F = -\oint_\infty \frac{dw}{2\pi i} w^{n-1} + \oint_\infty \frac{dz}{2\pi i} w(z)^n z^{-1}\,. \tag{B.7}$$

First terms integrates to zero and we get precisely the r.h.s. of eq. (B.5)!

Next, we want to show that for mixed derivatives of $\mathcal{B}$ are equal to those of $F(\boldsymbol{t})$. To do that, take n^{th} equation (5.6) and differentiate it with respect to $\bar{s}_k$:

$$\sum_{m=1}^{n} C_{n,m} \frac{\partial^2 \mathcal{B}(\boldsymbol{s}, \bar{\boldsymbol{s}})}{\partial \bar{s}_k \partial s_m} - k\,\bar{C}_{-n,-k} = 0\,. \tag{B.8}$$

Again, by the same reasons as before, we substitute $\partial_{t_k}\partial_{\bar{t}_n} F$ into (B.8) and rewrite it as

$$\oint_\infty \frac{dz}{2\pi i} w(z)^n \partial_{\bar{t}_k} D'(z) F = -k \oint_\infty \frac{dz}{2\pi i} z^{k-1} \left(\bar{w}(z)\right)^{-n}\,. \tag{B.9}$$

Rewrite eq. (B.1) in the form:

$$D(z)\bar{D}(\zeta) F = -\log\left(1 - \frac{1}{w(z)\bar{w}(\zeta)}\right) \tag{B.10}$$

(note, that we should have $|w(z)\bar{w}(\zeta)| > 1$ for the r.h.s. of this expression to be expansion in z^{-1}, ζ^{-1}). Now differentiate it with respect to z and extract the term, containing $\partial_{\bar{t}_k}$:

$$D'(z)\partial_{\bar{t}_k} F = -\oint_\infty \frac{d\zeta}{2\pi i} k\zeta^{k-1} \frac{w'(z)}{w(z)} \frac{1}{w(z)\bar{w}(\zeta) - 1} . \tag{B.11}$$

Now, substituting (B.11) into the l.h.s. of (B.9) we get

$$\oint_\infty \frac{d\zeta}{2\pi i} k\zeta^{k-1} \oint_\infty \frac{dw}{2\pi i} \frac{w^{n-1}}{w\bar{w}(\zeta) - 1} = \oint_\infty \frac{d\zeta}{2\pi i} \frac{k\zeta^{k-1}}{(\bar{w}(\zeta))^n} \tag{B.12}$$

(in the last equation we used the fact that $|w\,\bar{w}(\zeta)| > 1$). This result coincides with r.h.s. of (B.9), which proves the statement that mixed derivatives $\mathcal{B}(\boldsymbol{s}, \bar{\boldsymbol{s}})$ are equal to those of $F(\boldsymbol{t})$.

C. Homogeneity property of hermitean one-matrix model

We show in this section that partition sum of one matrix model, which is known to be a tau-function of KP hierarchy [11, 12] obeys certain homogeneity condition. Partition sum can be written as

$$\mathcal{Z}_{\text{herm}}(t) = \int \prod_{k=1}^{N} d\lambda_k \, \Delta^2(\lambda) \, e^{-NV(\lambda)} , \tag{C.1}$$

where $V(\lambda) = \sum_{k=1}^{\infty} t_k \lambda^k$ and $\Delta(\lambda)$ is a Van-der-Monde. One can see that $\mathcal{Z}_{\text{herm}}(N; \{t_k\}) = \mathcal{Z}_{\text{herm}}(Nt_1, Nt_2, \ldots)$. Then if one applies $\sum_{k=1}^{\infty} t_k \partial_{t_k}$ to $\mathcal{Z}_{\text{herm}}$ one gets:

$$\sum_{k=1}^{\infty} t_k \frac{\partial}{\partial t_k} \mathcal{Z}_{\text{herm}}(t) = \int \prod_{k=1}^{N} d\lambda_k \, \Delta^2(\lambda) \, e^{-NV(\lambda)} \left(N \sum_{k=1}^{\infty} t_k \lambda^k \right) . \tag{C.2}$$

This result can be also represented as $N \frac{\partial}{\partial_N} \mathcal{Z}_{\text{herm}}$. Now, it is well known (see e.g. [14]) that at large N partition sum of (C.1) should obey the genus g expansion:

$$\log \mathcal{Z}_{\text{herm}} = \sum_{g=0}^{\infty} N^{2-2g} F^{(g)}_{\text{herm}} . \tag{C.3}$$

So, for $N \to \infty$ property (C.3) implies: $N\partial_N \mathcal{Z}_{\text{herm}} = 2\mathcal{Z}_{\text{herm}}$. Thus, for the (logarithm of) partition sum of the hermitean one-matrix model one can get in the large N limit:

$$\sum_{k=1}^{\infty} t_k \frac{\partial F^{(0)}_{\text{herm}}}{\partial t_k} = 2F^{(0)}_{\text{herm}} . \tag{C.4}$$

This is precisely the scaling which we need in view of (3.10).

D. Homogeneity property of normal matrix model

One may wish to repeat the derivation of the appendix C for the case of 2D Toda tau-function F, given by the partition sum of normal matrix model [17]:

$$\mathcal{Z}_{\rm norm} = \int \prod_{k=1}^{N} d^2 z_k \, |\Delta_N(z)|^2 \, e^{-NV(z,\bar{z})} \,, \tag{D.1}$$

where $V(z,\bar{z}) = -z\bar{z} + \sum\limits_{k=1}^{\infty} (t_k z_k + \bar{t}_k \bar{z}_k)$. The term $z\bar{z}$ proves to make a significant difference. Namely, we cannot say that dependence on N for $\mathcal{Z}_{\rm norm}$ enters only in combinations Nt_k or $N\bar{t}_k$, there is also an explicit dependence on N. To get rid of it, one can rescale $z_k \to z_k/\sqrt{N}, \bar{z}_k \to \bar{z}_k/\sqrt{N}$ to

$$\mathcal{Z}_{\rm norm} \to \int \prod_{k=1}^{N} d^2 z_k \, |\Delta_N(z)|^2 \, \exp\left(-z\bar{z} + \sum_{k=1}^{\infty} \left(t_k N^{1-k/2} z_k + \bar{t}_k N^{1-k/2} \bar{z}_k\right)\right). \tag{D.2}$$

Repeating the reasonings similar to those of appendix C we are getting homogeneity condition (5.8) for F_0 — the leading term of the large N limit expansion of $\log \mathcal{Z}_{\rm norm}$:

$$\sum_{k=1}^{\infty} \left(1 - \frac{k}{2}\right) t_k \frac{\partial F_0}{\partial t_k} + \left(1 - \frac{k}{2}\right) \bar{t}_k \frac{\partial F_0}{\partial \bar{t}_k} + t_0 \frac{\partial F_0}{\partial t_0} = 2F_0 \tag{D.3}$$

(variable t_0 can be introduce in by means of: $\partial_{t_0} \mathcal{Z}_{\rm norm} = N\mathcal{Z}_{\rm norm}$). We see that the presence of $z\bar{z}$ terms makes the homogeneity condition quite different from simply two copies of (C.4).

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